Controlled simulations of high frequency markets : a Mean Field Game approach

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Plan of presentation

Introduction

Subject

simulate a trading day to test intra-day high frequency strategies

Remember

the backtesting ignores the market impact

Idea

use a two-scale model to simulate the dynamic of order books

Limit order book

- *best bid*, $b(t)$; *best ask*, $a(t)$
- *spread bid-ask*, $\psi(t) = a(t) - b(t)$
- *mid-price*, $p_m(t) = \frac{b(t)+a(t)}{2}$

Zero intelligence

- **Principle** : simulates the distributions of the ask, bid, spread, orders arrival, etc.
- **Example** :
  - market orders randomly arrive with a Poisson rate of μ shares per unit time
  - limit orders arrive at a distance d from the opposite best quote at independent, exponential times with rate $\lambda(d)$
  - limit orders are cancelled according to a Poisson process, with a fixed rate θ per unit time (here the orders arrival)
- **Remark** : A "zero intelligence" model focuses on some characteristics of the market selected before (here the orders arrival). When the simulation is done we find back these characteristics, but we do not necessarily find some other important market characteristics (for exemple the correlation between price and order flows).

Results and first Conclusions

Result

this method gives extreme values for prices

Summary

this approach is purely phenomenological, the market mechanisms are considered as physical systems and consequently disregard the economic dimension, i.e. the mechanisms of "rational" decisions. Nevertheless it's really the decision-making process which explains the dynamics of LOB and consequently the market evolution.

Conclusion

the need for a "real" decision-making process at the macroscopic level is real

Principle

- we propose a two-scale model to simulate the dynamic of order books :

Macroscopic scale

a decision-making process on a macroscopic scale that describes the theoretical dynamic view of the agents on the price

Microscopic scale

a microscopic scale, similar to "zero intelligence" models, which according to the view of the agents on the price and conditionally to the current LOB describes the dynamic of the LOB

Plan of presentation

Principle

Questions

1. how this theoretical macroscopic model of the LOB “decides” the microscopic dynamic of the LOB ?
2. how to send “information” from the macroscopic to the microscopic level ?

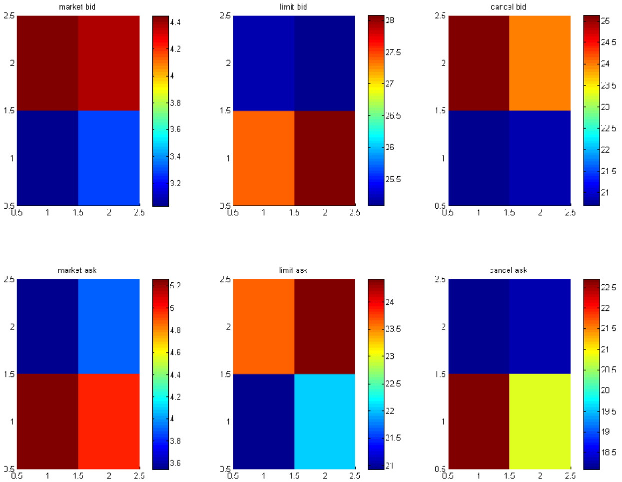
Alternative

for statistical reasons we chose to use two parameters : the price p^* and the slope of the LOB at the mid-price $\ell(t)$

Empirical laws

- at time t we consider the current price $p(t)$ and the current slope $\ell(t)$
- we consider the price $p^{(n)}(t)$ and the slope $\ell^{(n)}(t)$ after n trades
- we compute the empirical laws of $\Delta p = p^{(n)}(t) - p(t)$ and $\Delta\ell = \ell^{(n)}(t) - \ell(t)$
- we compute the empirical law of order types conditionally to Δp and $\Delta\ell$
- we choose n in order to maximize a criterion (for exemple the Kullback-Leibler divergence)

Order types conditionally to Δp and $\Delta \ell$



Microscopique scale (I)

Step 1

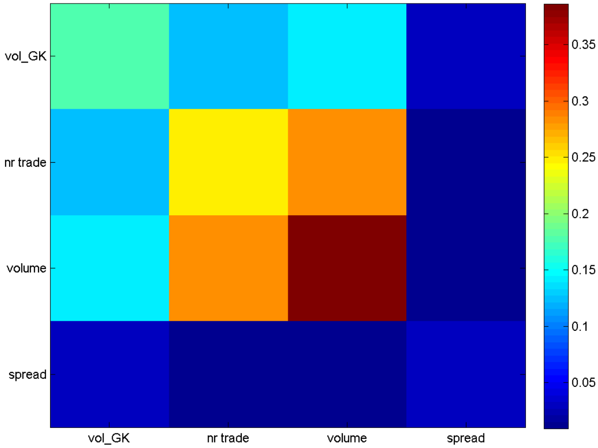
in order to take into account the specificity of each title we considered a log-normal model :

$$\log(V, \psi, \sigma, N) \sim \mathcal{N}(\mu, \Sigma)$$

where

- the cumulative volume for the period δt =10 minutes, V
- the number N of transactions on the period δt
- the average spread per transaction, ψ
- the 10-minute Garman-Klass volatility, σ

Covariance matrix



Microscopique scale (II)

Step 2

given a σ, we simulate the theoretical macroscopic prices and slopes (input parameter)

Step 3

at every moment we empirically decide the arrival time of the next order, δt

Step 4

conditionally to Δp and $\Delta \ell$ we decide the order type

Microscopique scale (III)

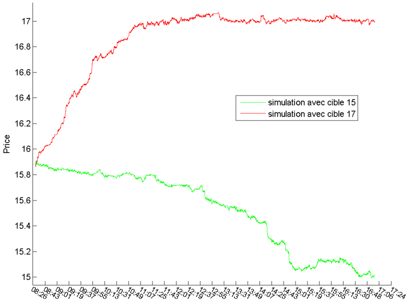
Step 5

we compute the conditional distribution of $\log(V, \psi, \sigma, N)$ with respect to σ and $N \approx \frac{1}{\delta t}$

Step 6

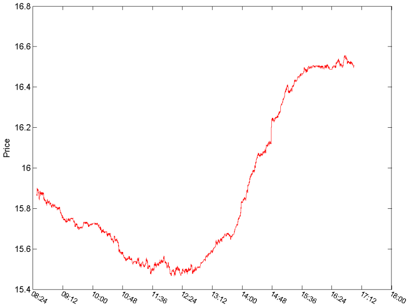
if we have a limit or a cancel order, conditionally to ψ we decide at which price we send the order to the market

Simulations (1)



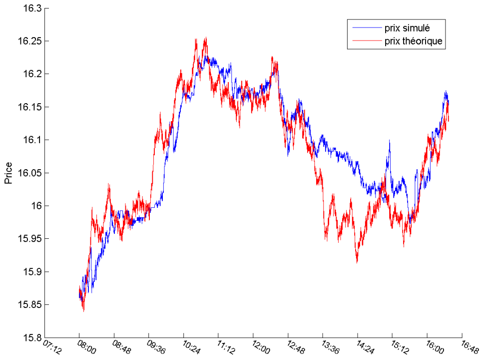
simulations with a starting price corresponding to the day of FTE.PA 15/07/2009 and a constant price target 15 and respectively 17

Simulations (2)



simulation of a price trajectory with a price target at 15.5 during the first half of the day and a new target at 16.5 for the second half of the day.

Simulations (3)



simulation (blue) with a price that “follows” a theoretical price path (in red), here a brownian motion

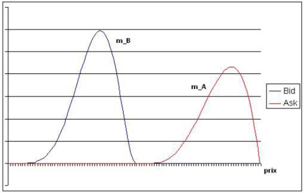
Plan of presentation

Presentation of MFGen

- MFGen simulates time series of prices and LOB slopes using a model inspired from the mean field games theory.
- To model the dynamics of limit order books, we use a transport equation which involves the behaviors of strategic agents. These behaviors are not obtained using a Bellman equation but rather heuristic rules
- There are three types of agents : noise traders, trend followers, mean reverters.

The underlying ideas (I)

A LOB can be represented by two functions m_B and m_A :



If we consider separately twe two sides of the market (hence ignoring executions), the evolution of the LOBs can be modeled by two heat equations to take account of the changes in opinions.

$$\partial_t m_B(t,p) - \frac{\epsilon^2}{2}\partial^2_{pp} m_B(t,p) = 0$$

$$\partial_t m_A(t,p) - \frac{\epsilon^2}{2}\partial^2_{pp} m_A(t,p) = 0$$

The underlying ideas (II)

The two LOBs are not independent and if p^* denotes the equilibrium price, demand and supply flows must be equal :

$$\partial_p m_B(t, p^*(t)) = -\partial_p m_A(t, p^*(t))$$

We introduce m :

$$m(t,p) = \begin{cases} m_A(t,p), & \text{if } p \geq p^* \\ -m_B(t,p), & \text{if } p < p^* \end{cases}$$

and $p^*(t)$ is now implicitly defined by $m(t, p^*(t)) = 0$.

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Reintroduction of orders (I)

- Executed orders must be reintroduced in the market.
- A simple way to reintroduce orders is to consider noise traders :

$$\partial_t m(t,p) - \frac{\epsilon^2}{2}\partial^2_{pp} m(t,p) = -\partial_p m(t, p^*(t))\, m(t,p)$$

- More generally, we are considering that agents have different strategies :
  - Buyers (resp. Sellers) can be trend followers of horizon T if the price was lower (res. higher) T periods ago ($p^*(t-T) < p^*(t)$, resp. $p^*(t-T) > p^*(t)$)
  - Buyers (resp. Sellers) can be mean reverters of horizon T if the T-period moving average was lower (res. higher) than the current price ($\frac{1}{T}\int_{t-T}^{t} p^*(s)ds < p^*(t)$, resp. $\frac{1}{T}\int_{t-T}^{t} p^*(s)ds > p^*(t)$)
  - Buyers and sellers can be noise traders

Reintroduction of orders (II)

- Three types of agents : trend-followers, mean-reverters, noise traders
- Time horizons T are Gamma-distributed
- Reintroductions are made using a symmetry rule, both for mean-reverters and trend followers
- **Types of traders are determined ex-post** just to decide where to reintroduce orders.
- The equation for m is now :

$$\partial_t m(t,p) - \frac{\epsilon^2}{2}\partial^2_{pp} m(t,p) = -\partial_p m(t,p^*(t)) \left[source(t,p)\right]$$

where the source function is determined using the process described above.

Introduction of noise

- We need to introduce noise in the model
- Noise will be on new orders or removed orders :

$$dm(t,p) = \left[\frac{\epsilon^2}{2}\partial^2_{pp} m(t,p) - \partial_p m(t, p^*(t)) \left[source(t,p)\right]\right] dt$$

$$+\nu m(t,p) g(p, p^*(t)) dB^p_t$$

where brownian motions are independent and where $p \mapsto g(p, p^*(t))$ is equal to zero for $p = p^*(t)$ (+ boundary conditions).
(In reality, the noise is capped)

Remarkable properties (I)

convexity property

The evolution of the price is linked to the **convexity** of the order book !

$$\dot{p}(t) = -\frac{\epsilon^2}{2} \frac{\partial^2_{pp} m(t, p^*(t))}{\partial_p m(t, p^*(t))} dt$$

proof

Let's differentiate $m(t, p^*(t)) = 0$:

$$dp^*(t) = -\frac{\partial_t m(t, p^*(t))}{\partial_p m(t, p^*(t))} dt$$

- It corresponds to real features.

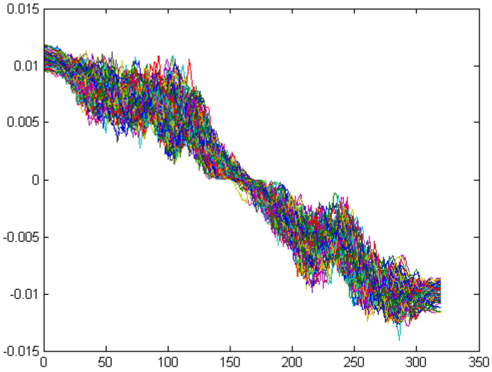
Remarkable properties (II)

volatility properties

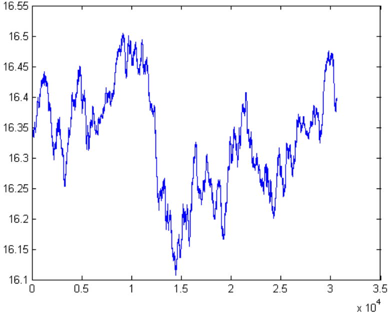
- σ_p is an increasing function of ν and an increasing function of ϵ.
- σ_ℓ is an increasing function of ν and a decreasing function of ϵ (smoothing effect).

- The output of the algorithm is the volatility of prices (σ_p) and the "volatility" of slopes (σ_ℓ).
- Calibration of the model will be possible due to these results.

Example (LOB dynamics)



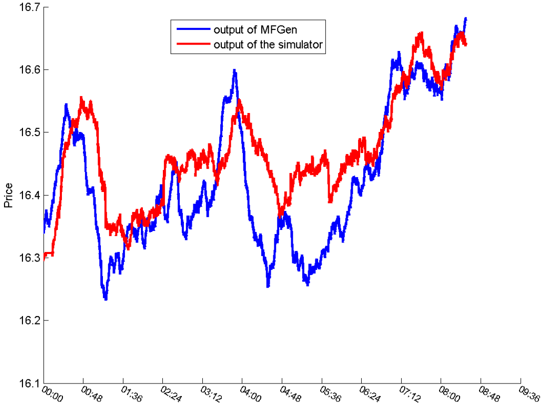
Example (prices)



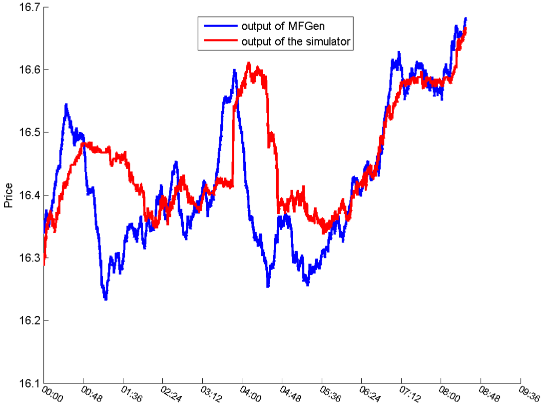
For some values of ϵ and ν we obtain a volatility of 23.7%. The calibration on actual values of slopes is still at stake, the relevant observation (standard deviation, ...) to calibrate upon still being a matter of debate.

Plan of presentation

Example (1)


output of MFGen
output of the simulator
Price
16.7
16.6
16.5
16.4
16.3
16.2
16.1
00:00
00:48
01:36
02:24
03:12
04:00
04:48
05:36
06:24
07:12
08:00
08:48
09:36

Example (2)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output of MFGen
output of the simulator

Remarks

	AM	AC	BL	BM	BC	AL	Difference	Pcb
FTE.PA	4.18%	22.10%	23.86%	4.53%	20.32%	25.12%	$6.6 * 10^{-4}\%$	11,36%
cible 17	5.46%	20.65%	24.56%	4.69%	21.63%	22.98%	1.40%	8.57%
cible 15	5.16%	19.72%	23.02%	5.38%	22.20%	24.53%	−4.20%	10.33%
cible 15.5 − 16.5	5.69%	20.74%	24.72%	4.27%	21.66%	22.92%	2.30%	8.60%
cible "brown"	5.35%	20.44%	23.81%	5.26%	21.75%	23.39%	−0.81%	9.23%
Simulation 1	4.23%	19.80%	26.15%	3.98%	23.47%	22.30%	0.36%	10.01%
Simulation 2	4.77%	19.71%	25.49%	4.64%	23.25%	22.14%	$5.8 * 10^{-4}\%$	8.10%

- A = ask B = bid
- M = market C = cancel L = limit
- Difference = (AM+AC+BL)-(BM-BC-AL)