Limit Order Book and MFG. A first stone.

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Abstract

This article the first stone of a mean field game approach to market microstructure. The goal of this first draft is to exhibit a way to model LOBs with a continuous setting on which one can build behaviors of rational agents. The model presented must be seen as a microstructure-level model of noise traders. An important caveat that will be solved in the future is the absence of bid/ask spread.

Introduction and notations

We will consider a diffusion model for the orders on each side of the market. The density of orders will be denoted $m_A(t,p)$ and $m_B(t,p)$, the indices meaning Ask and Bid respectively. Hence, the number of (unitary) buying orders between p and $p+dp$ is $m_B(t,p)dp$.
Now, since we face a diffusion model, there is a flow of order going through the price between t and $t+dt$. These (signed) flows are denoted $j_A(t,p)dt$ and $j_B(t,p)dt$ depending on the side of the book (Bid or Ask).
Since we consider a continuous market, there must be an equilibrium at each time t. Therefore, if one denotes p^* the equilibrium price, we must have:

$$\boxed{j_B(t,p^*) = -j_A(t,p^*) \qquad m_B(t,p^*) = m_A(t,p^*) = 0}$$

Hence, if we consider that the diffusion parameter is σ, the classical relation $j(t,p) = -\frac{1}{2}\partial_p(\sigma(t,p)m(t,p))$ leads to $\partial_p m_B(t,p^*) = \partial_p m_A(t,p^*)$, assuming a regular function for σ. This is interesting since we are then able to consider the regular function m defined by m_B on the Bid side and defined by $-m_A$ on the Ask side.

The model

We consider a diffusion model. Hence, if we consider the Bid side for instance, the dynamics should be given (assuming σ is a constant) by a heat equation like $\partial_t m(t,p) - \frac{\sigma^2}{2}\partial^2_{pp} m(t,p) = 0$. However, two questions appear naturally : What should we do with the executed order? What should we do with orders that go

outside the maximum range of price $[\underline{P}, \overline{P}]$?
To answer these questions we have to make some hypotheses. First, we do not want to loose mass. Hence executed orders should be reintroduced. If someone buy a stock (bid side), it's quite normal to introduce (perhaps far above the current equilibrium price) a selling order (ask side). Our hypothesis is therefore to reintroduce any bid order that is executed or goes out of the range on the ask side and symmetrically. To reintroduce the orders, we use a distribution function g_A (resp. g_B) whose mass is supported by the Ask side (resp. the Bid side).
Hence the basic model is:

$$\partial_t m_B(t,p) - \frac{\sigma^2}{2}\partial^2_{pp} m_B(t,p) = g_B(t,p)\left(-j_A(t,p^*) + j_A(t,\overline{P})\right)$$

$$\partial_t m_A(t,p) - \frac{\sigma^2}{2}\partial^2_{pp} m_A(t,p) = g_A(t,p)\left(j_B(t,p^*) - j_B(t,\underline{P})\right)$$

The problem to solve is therefore (with $m(0,\cdot)$ given):

$$\partial_t m(t,p) - \frac{\sigma^2}{2}\partial^2_{pp} m(t,p) = g_B(t,p)\left(-\frac{\sigma^2}{2}\partial_p m(t,p^*) + \frac{\sigma^2}{2}\partial_p m(t,\overline{P})\right)$$

$$+ g_A(t,p)\left(-\frac{\sigma^2}{2}\partial_p m(t,p^*) + \frac{\sigma^2}{2}\partial_p m(t,\underline{P})\right)$$

Some examples are below:

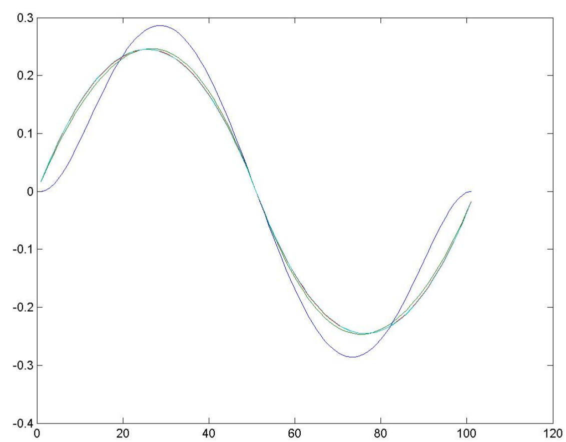

Figure 1: Symmetrical Beta Law for g

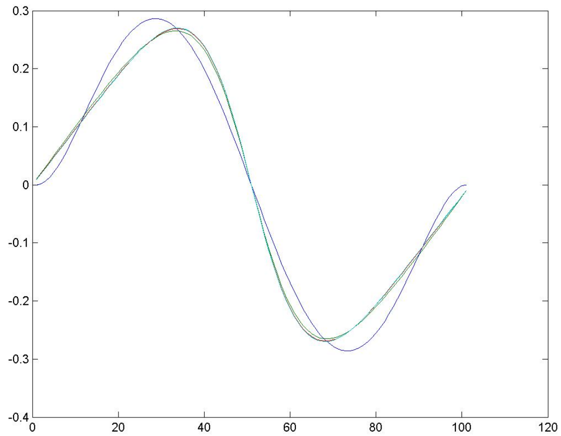

Figure 2: Non-Symmetrical Beta Law for g

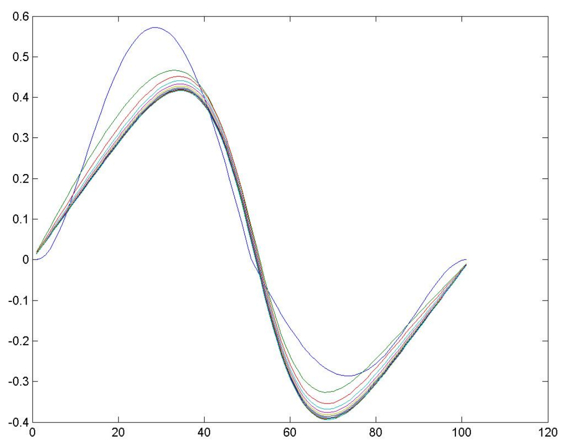

Figure 3: Upward shift due to a higher mass on the bid side

Now, we are going to introduce randomness in this setting and obtain a stochastic partial differential equation. The randomness is due to the fact that more or less orders are added at a given price (idiosyncratic noise) or it can be due to a global mass disequilibrium between the two sides.

To model this noise we generalize the preceding equation:

$$dm(t,p) = \left[\frac{\sigma^2}{2}\partial^2_{pp}m(t,p) = g_B(t,p)\left(-\frac{\sigma^2}{2}\partial_p m(t,p^*) + \frac{\sigma^2}{2}\partial_p m(t,\overline{P})\right)\right.$$

$$\left.+g_A(t,p)\left(-\frac{\sigma^2}{2}\partial_p m(t,p^*) + \frac{\sigma^2}{2}\partial_p m(t,\underline{P})\right)\right] dt + \nu m(t,p)dW^p_t + \varepsilon m(t,p)dW_t$$

This gives:

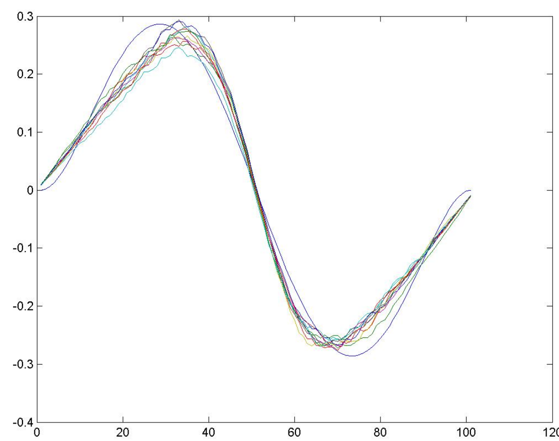

Figure 4: Case $\nu = 1$

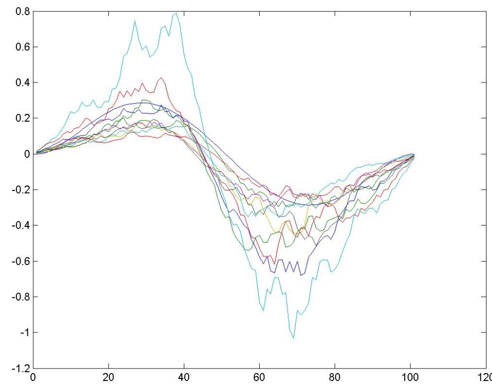

Figure 5: Case $\nu = 5$

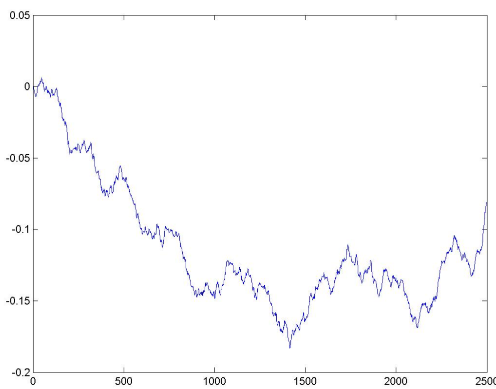

Figure 6: Price move associated to the $\nu = 5$ case

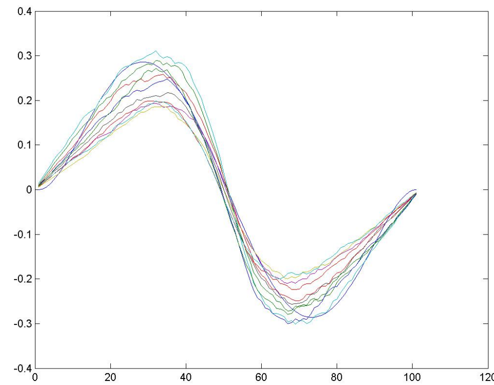

Figure 7: Case $\nu = 1$ and $\epsilon = 1$

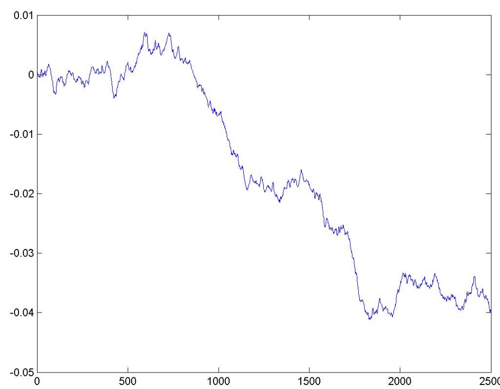

Figure 8: Price move associated to the case $\nu = 1$ and $\epsilon = 1$

We see that such an equation can easily lead to price movements that resemble real prices.

To end with the different possibilities, one can also consider a function for σ instead of a constant and in that case the equations are a bit changed:

$$\partial_t m(t,p) - \frac{1}{2}\partial^2_{pp}(\sigma^2(t,p)m(t,p)) = g_B(t,p)\left(-\frac{1}{2}\partial_p(\sigma^2(t,p^*)m(t,p^*)) + \frac{1}{2}\partial_p(\sigma^2(t,\overline{P})m(t,\overline{P}))\right)$$

$$+g_A(t,p)\left(-\frac{1}{2}\partial_p(\sigma^2(t,p^*)m(t,p^*)) + \frac{1}{2}\partial_p(\sigma^2(t,\underline{P})m(t,\underline{P}))\right)$$

An example is given below:

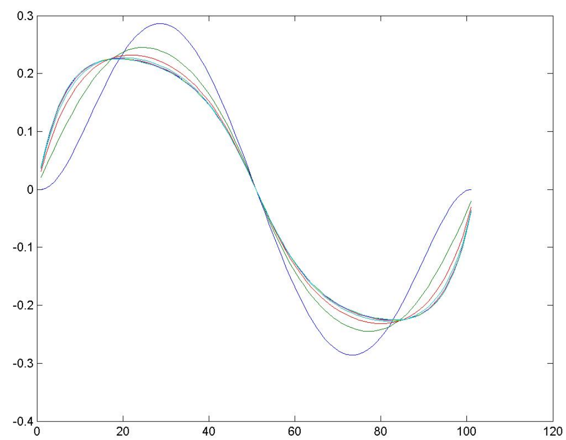

Figure 9: Case $\sigma^2 = \exp(-(p-p^*)^2)$

Concluding remarks

Our model is rich enough to model noise traders and one can build a mean field game upon this model. However, calibration is needed and one can see that the calibration is not a real difficulty as soon as σ is a constant. In the equation,

$$dm(t,p) = \left[\frac{\sigma^2}{2}\partial^2_{pp}m(t,p) = g_B(t,p)\left(-\frac{\sigma^2}{2}\partial_p m(t,p^*) + \frac{\sigma^2}{2}\partial_p m(t,\overline{P})\right)\right.$$

$$\left.+g_A(t,p)\left(-\frac{\sigma^2}{2}\partial_p m(t,p^*) + \frac{\sigma^2}{2}\partial_p m(t,\underline{P})\right)\right] dt + \nu m(t,p)dW^p_t + \varepsilon m(t,p)dW_t$$

the noise terms can be extracted easily since they are respectively idiosyncratic and global noise. σ^2, g_A and g_B can then be calibrated using the exchanged volume and the form of the LOBs.